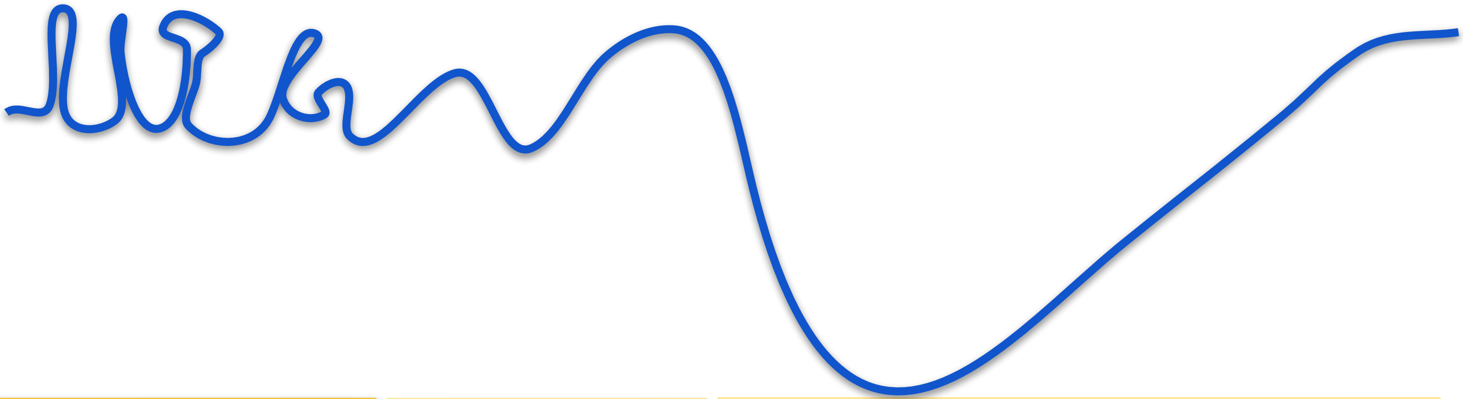
Computing with Signals



DA 623

Jan - May 2023

IIT Guwahati

Instructors: Neeraj Sharma

Lecture-05

Signal → Model *or* Representation → Processing


Signal
Model
or
Representation
Processing

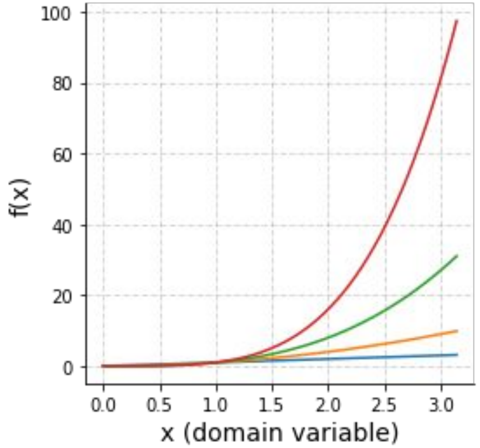

Signal
Model
or
Representation
Processing

- Polynomial representation

- Polynomial representation
- Fourier representation

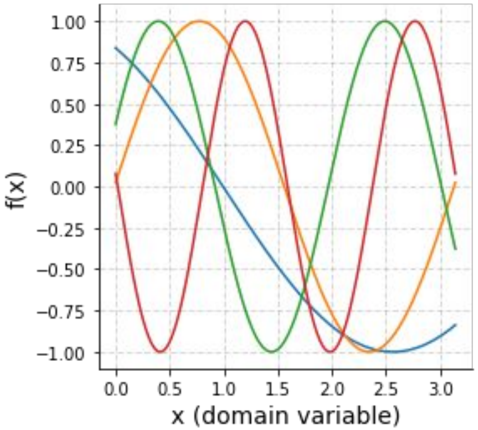
- Polynomial representation

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= \sum_{m=0}^{\infty} a_m x^m$$



- Fourier series representation

$$f(x) = \sum_{m=0}^{\infty} A_m \sin\left(\frac{\pi m x}{L} + \phi_n\right)$$



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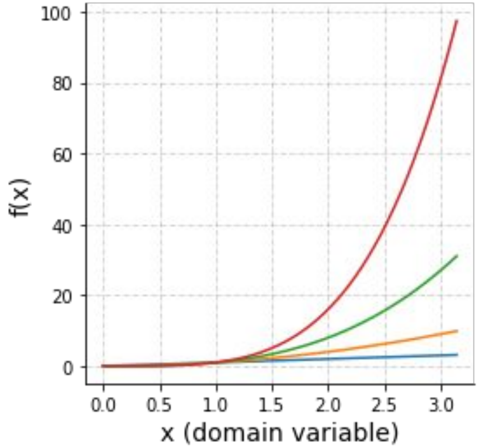

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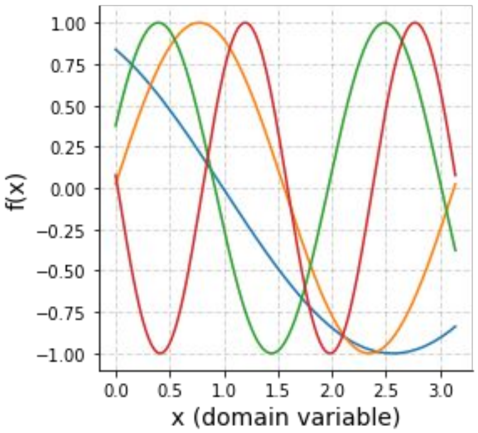
Focus of this lecture



- Fourier series representation

$$f(x) = \sum_{m=0}^{\infty} A_m \sin\left(\frac{\pi m x}{L} + \phi_n\right)$$

sin(A + B) = sin(A).cos(B) + cos(A).sin(B)

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{\pi m x}{L}\right) + b_m \sin\left(\frac{\pi m x}{L}\right)$$

A function can be written as sum of scaled cosine() and sine() functions

Jean-Baptiste Joseph Fourier
French Mathematician & Physicist
(1768 - 1830)

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On the Eiffel Tower, 72 names of French scientists, engineers, and mathematicians are engraved in recognition of their contributions.

Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- It is sum of many (infinite) terms
- Each of the cos(.) and sin(.) term is periodic - 2L/m
- Parameters of the sum are $\{a_o , a_m , b_m\}$
- It is a linear summation of cos(.) and sin(.), with no cross-terms

Fourier series representation

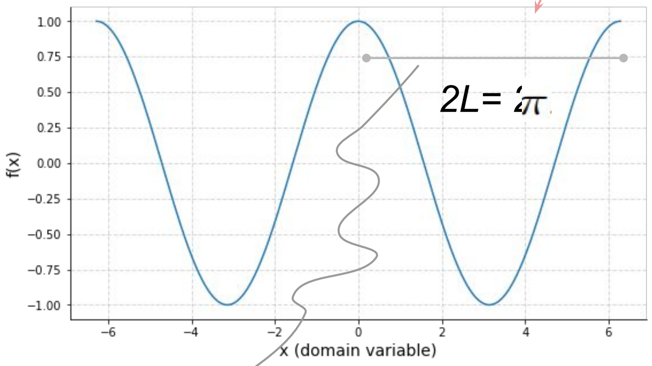
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

Let's visualize the cosine term

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

Example: L = π

m=1



This function has a period of 2L

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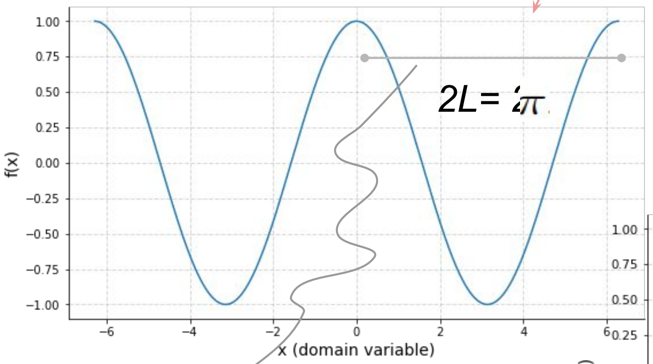
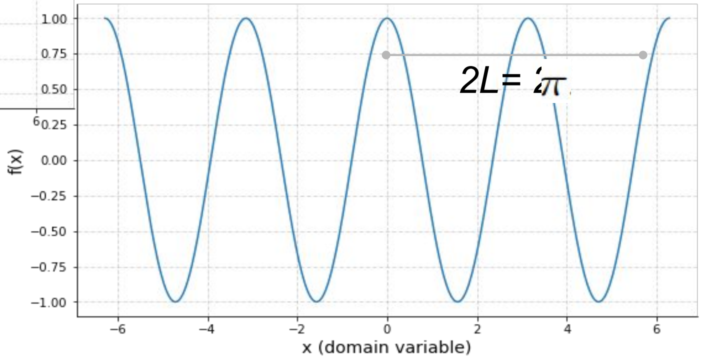


This function has a period of 2L

m=2



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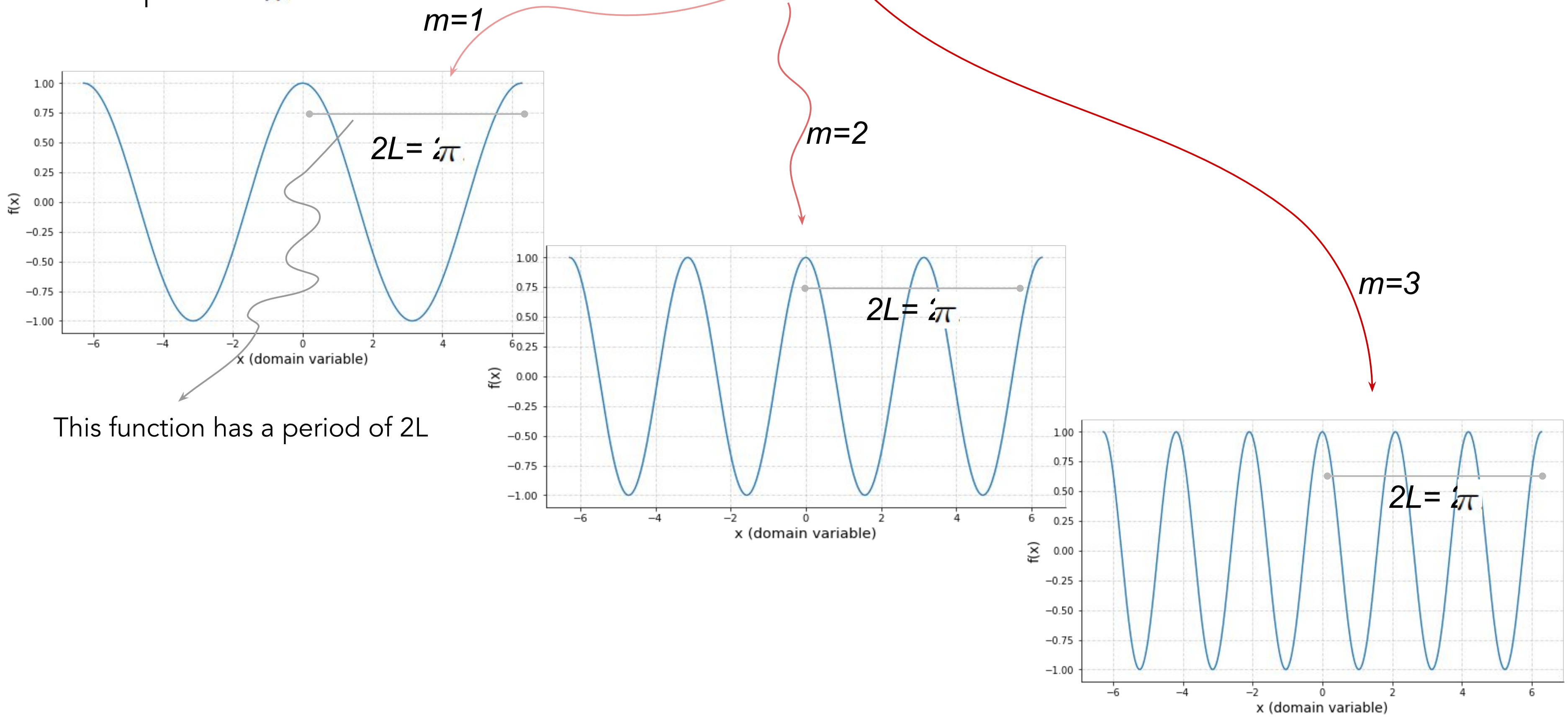

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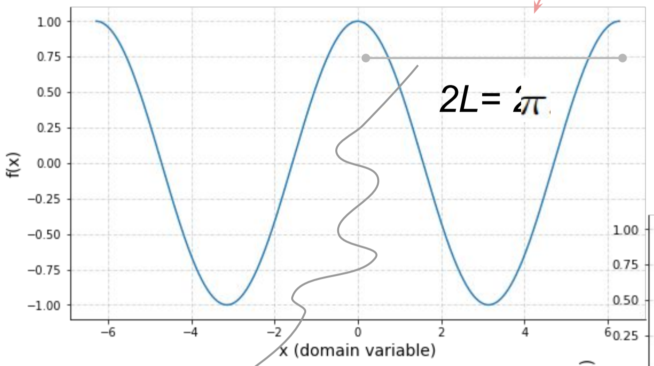
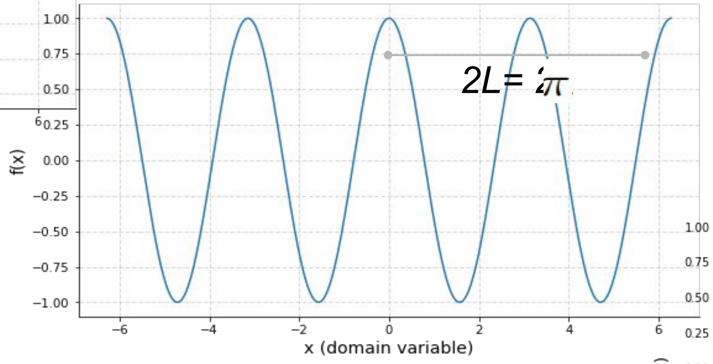
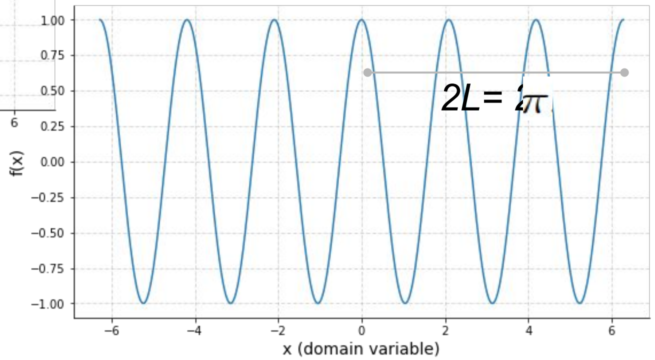


This function has a period of 2L

m=2

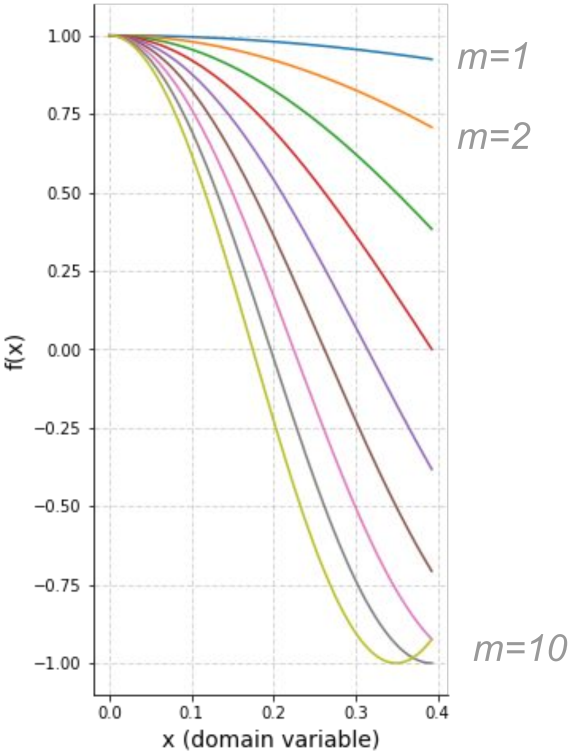


m=3



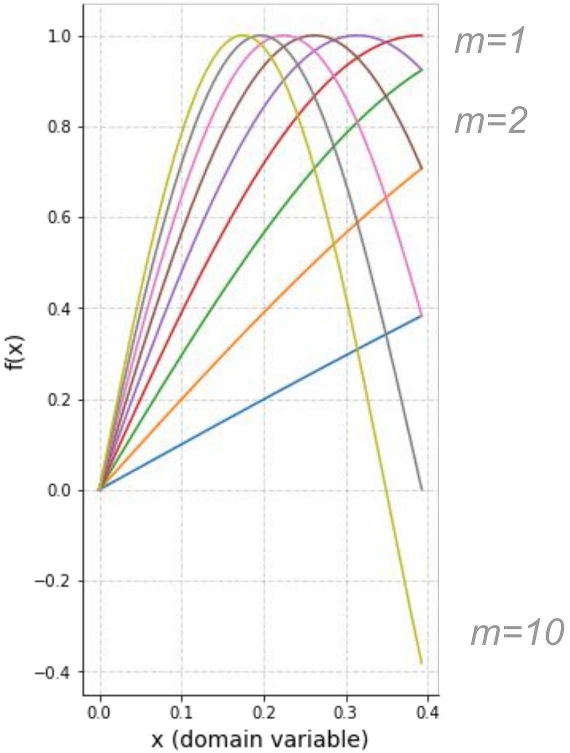
increasing m implies increasing cycles within 2L

$$\cos(\frac{m\pi x}{L})$$



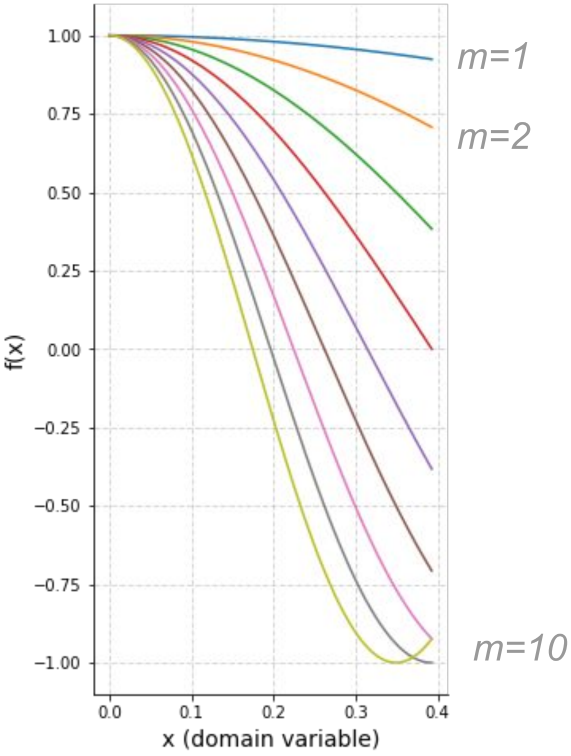
Spans only even functions of period 2L

$$\sin(\frac{m\pi x}{L})$$



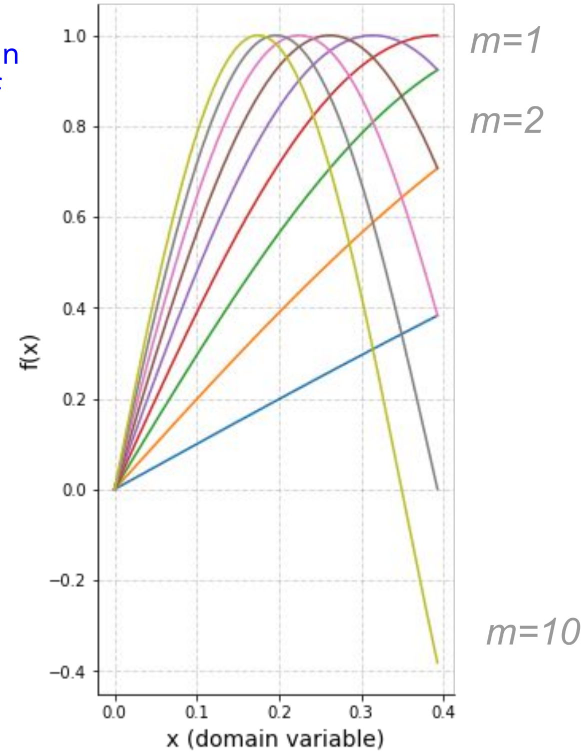
Spans only odd functions of period 2L

$$\cos\left(\frac{m\pi x}{L}\right) \quad + \quad \sin\left(\frac{m\pi x}{L}\right)$$



Spans only even functions of period 2L

Together they span ~ all functions of period 2L



Spans only odd functions of period 2L

Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- Consider a 2L periodic signal f(x)
- How do we compute $\{a_o , a_m , b_m\}$ to represent f(x) in the above form?

Let's first review some properties of sine and cosine functions. This will help us.

Fourier series representation

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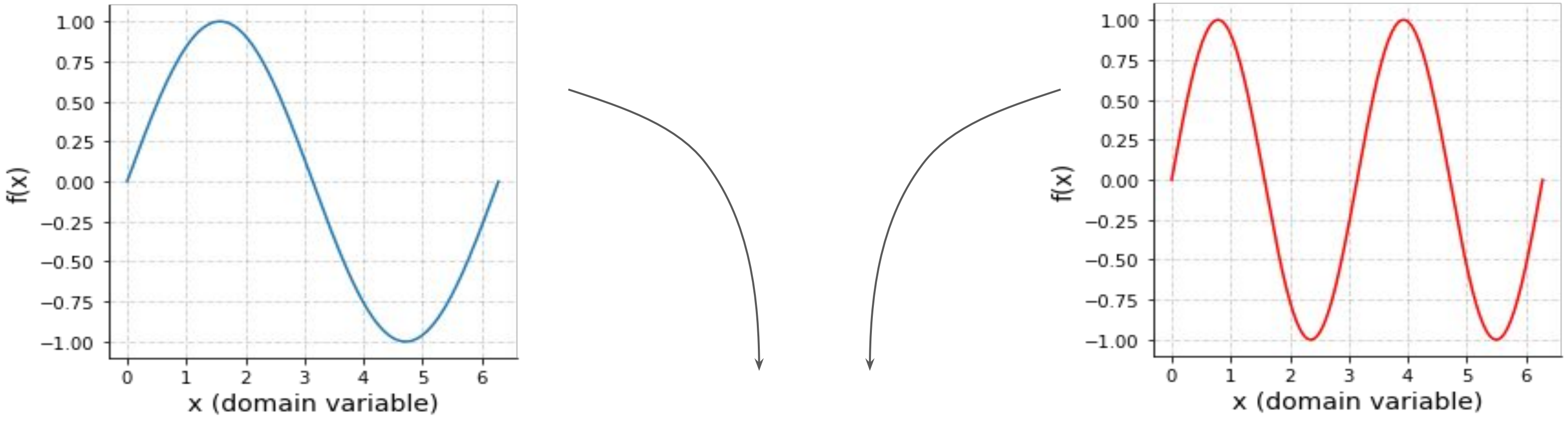


$$\int_0^{2L} f(x) = 0$$

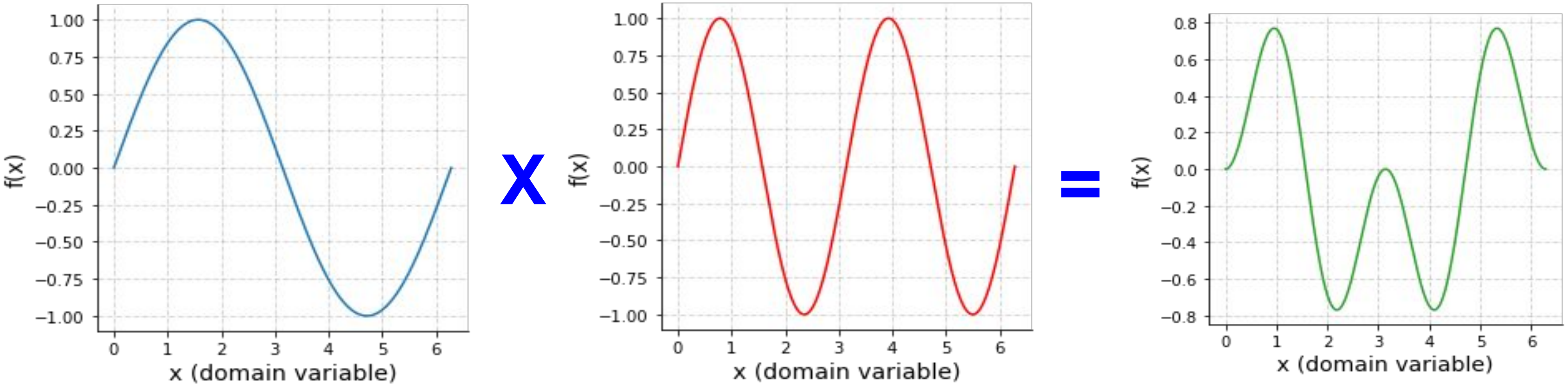
Integration of sin(.) function over a period (or its multiples) is 0.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- Integration of sin(.) function over a period (or its multiples) is 0.
- The same holds for cosine(.) as well.
- Thus, by integrating both sides of the above equation, we get

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

Let’s see a cross-term, that is multiplication of two sin(.)



$$\int_0^{2L} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) = 0$$

Integration of cross-terms over a period (or its multiples) is 0.

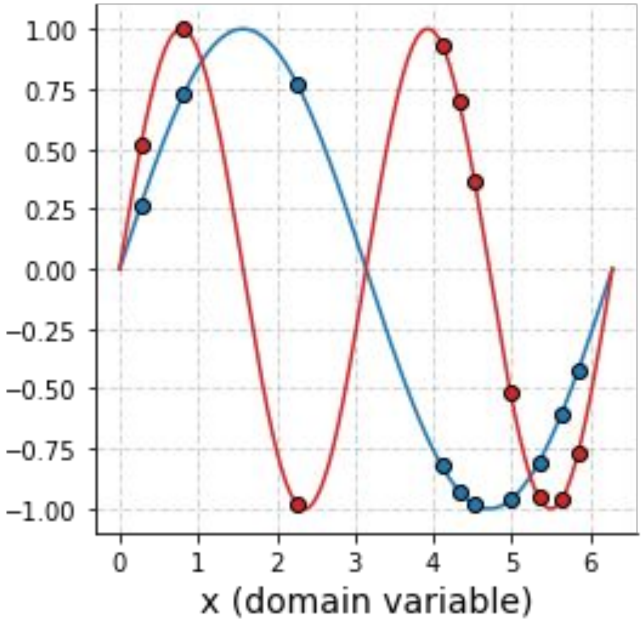
Further, integration of any two cross-terms over a period (or its multiples) is 0.

$$\int_0^{2L} \sin\left(\frac{m_1 \pi x}{L}\right) \sin\left(\frac{m_2 \pi x}{L}\right) = 0, \; m_1 \neq m_2$$

$$\int_0^{2L} \cos\left(\frac{m_1 \pi x}{L}\right) \cos\left(\frac{m_2 \pi x}{L}\right) = 0,$$

$$\int_0^{2L} \sin\left(\frac{m_1 \pi x}{L}\right) \cos\left(\frac{m_2 \pi x}{L}\right) = 0$$

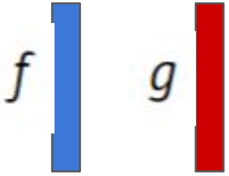
The sin(.) and cosine(.) functions as used in Fourier series are orthogonal functions.

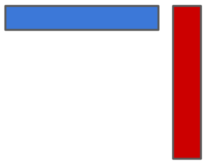


Familiar with **orthogonal vectors** in Euclidean spaces

Visualize the vectors as composed of values sampled from functions.

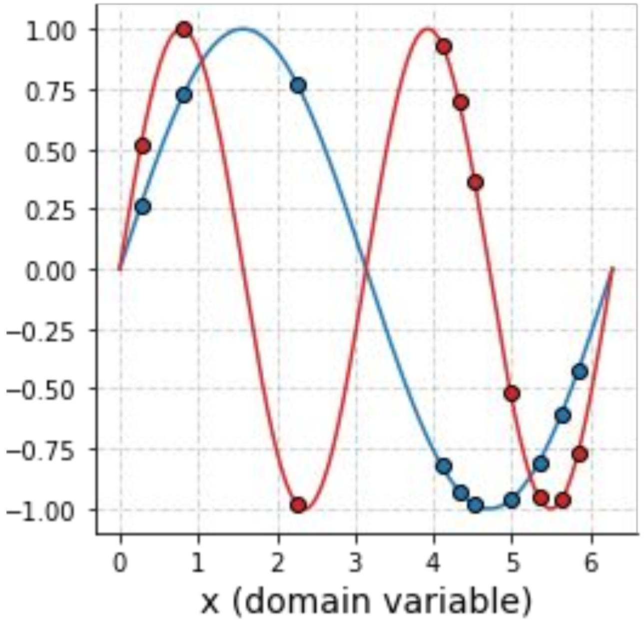
Then, orthogonality implies,

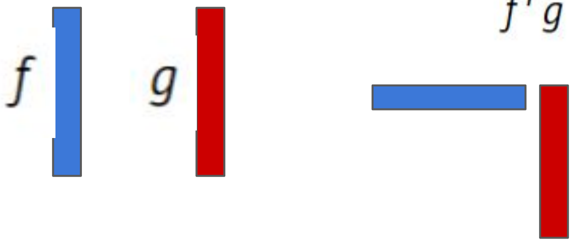




$$f^T g = \sum_{k=1}^{n} f[k]g[k] = 0$$

The sin(.) and cosine(.) functions as used in Fourier series are orthogonal functions.





$$f^T g = \sum_{k=1}^{n} f[k]g[k] = 0$$

Related by Riemann integral approximation.

$$< f, g > = \int_0^{2L} f(x)g(x)dx = 0$$

For functions, the inner product is equal to 0.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- Thus, by multiplying both sides by corresponding sin() or cosine() term and integrating, we get

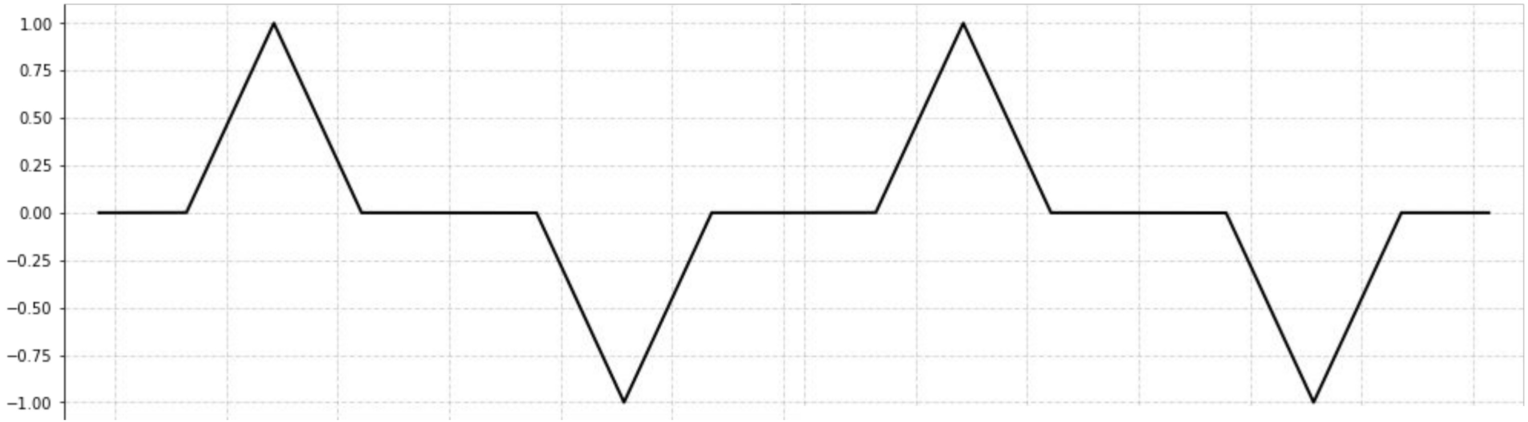
$$\begin{aligned} a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{m\pi x}{L}) dx, \text{ and} \\ b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{m\pi x}{L}) dx. \end{aligned}$$

Summary,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{m\pi x}{L})dx, \text{ and} \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})dx. \end{aligned}$$

Consider the signal,



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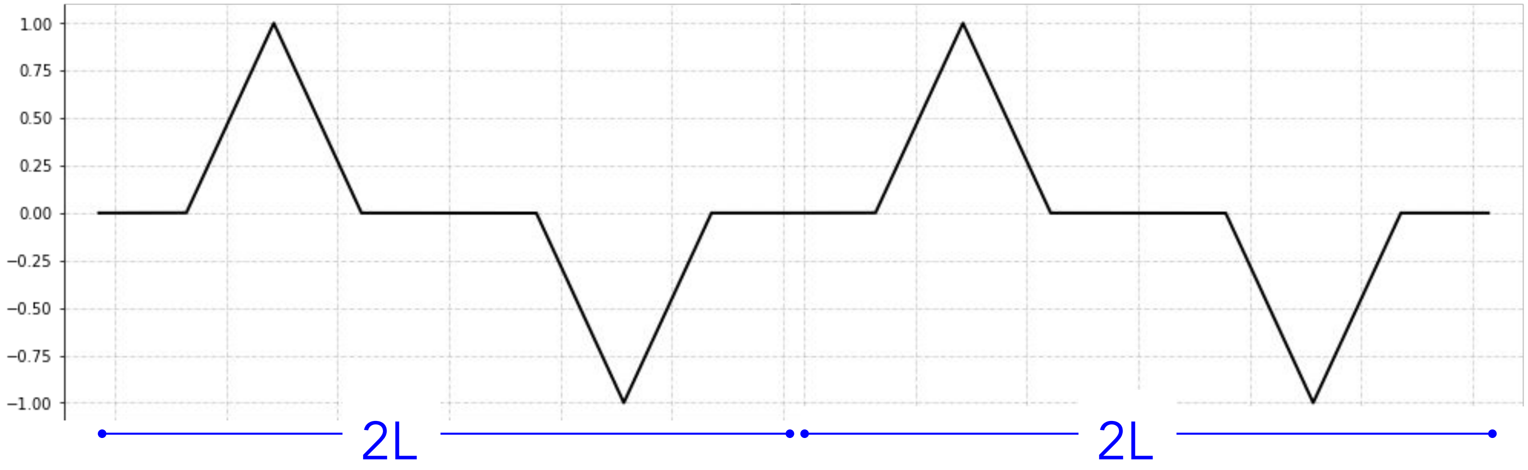
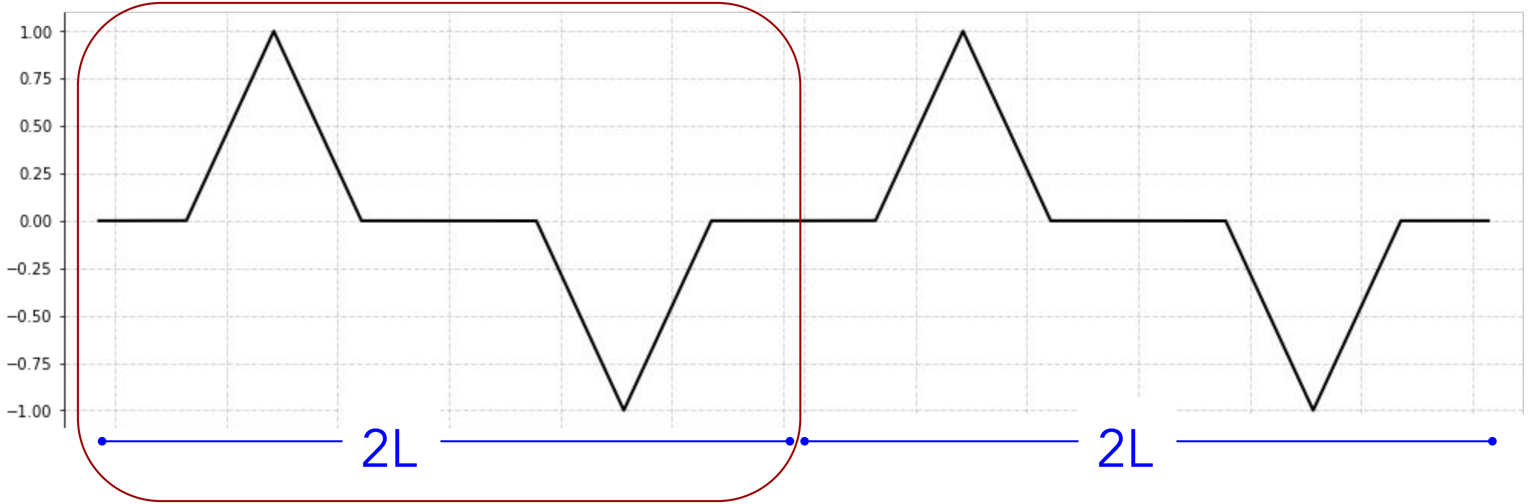


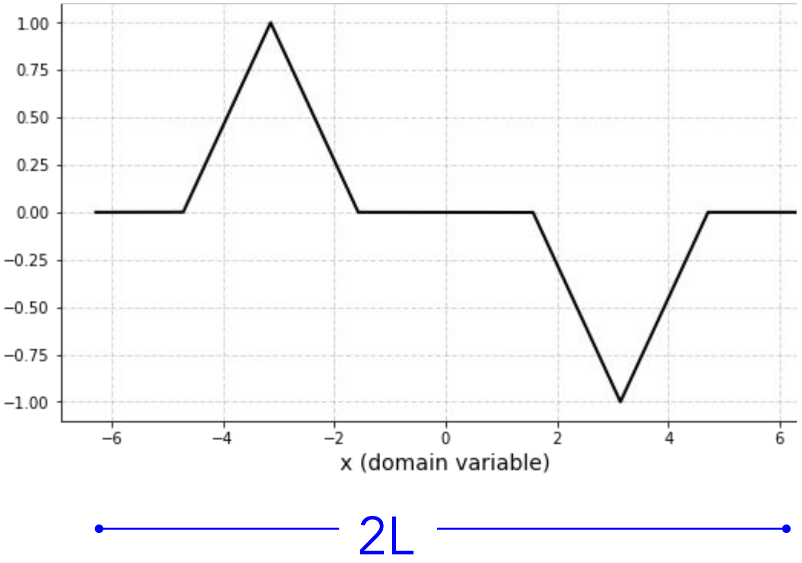
Consider the periodic signal,

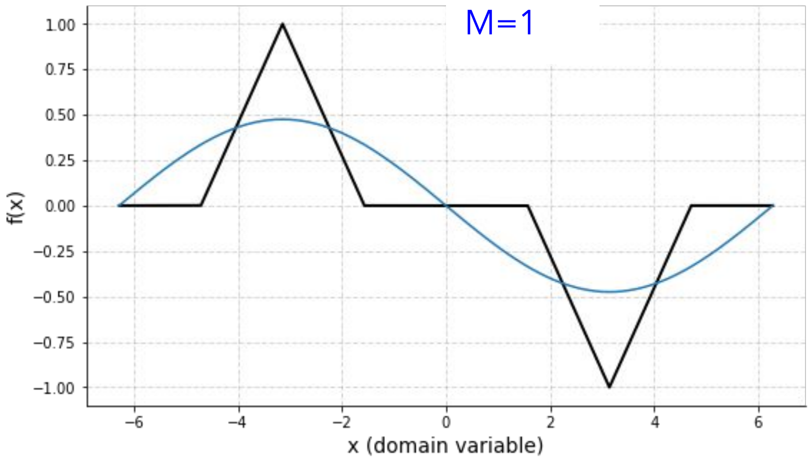


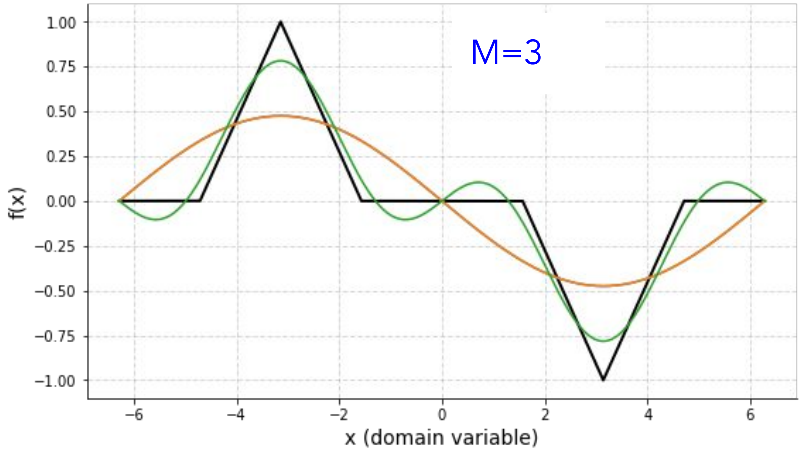
Can we express this signal using a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

Consider the periodic signal,

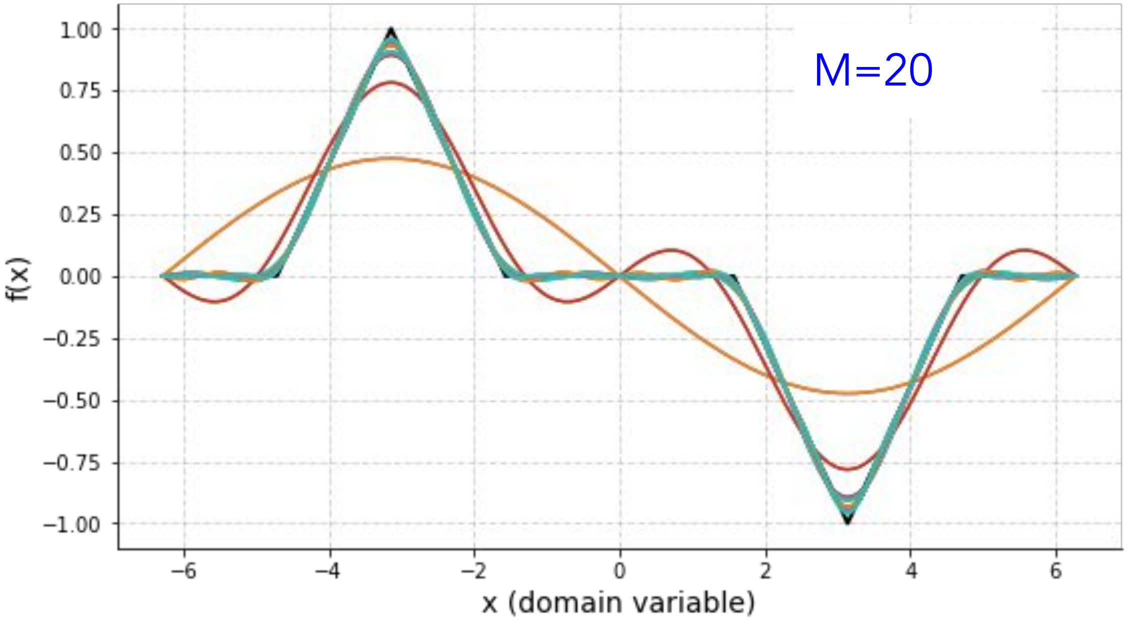


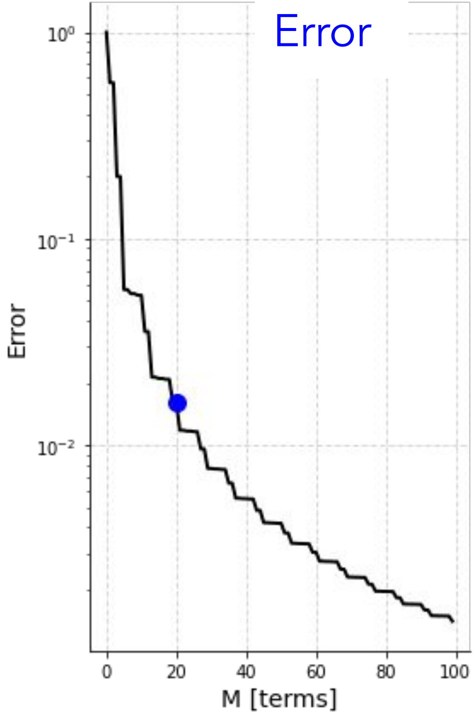




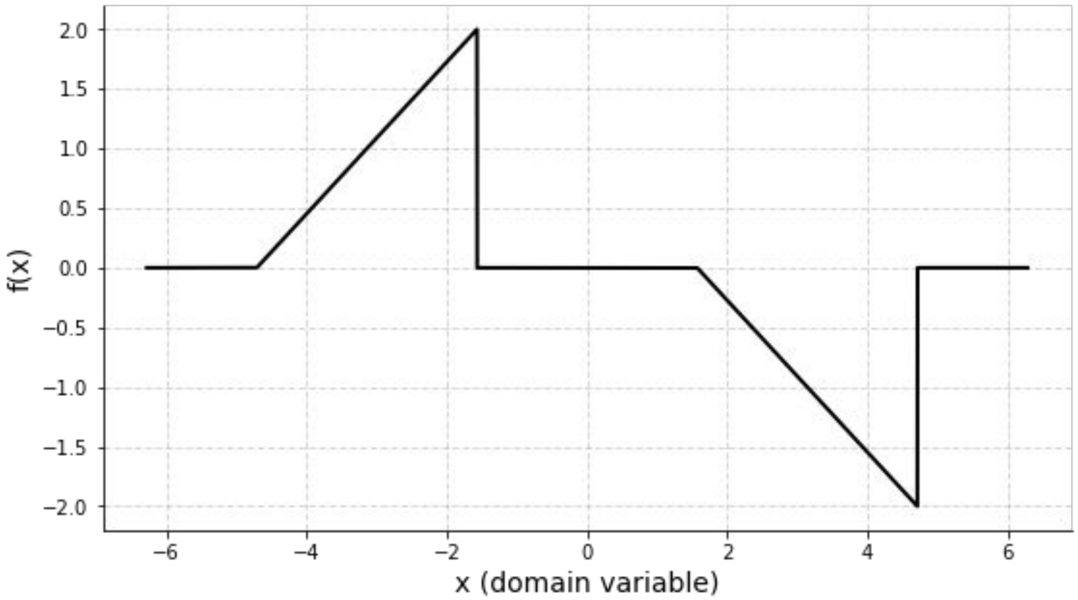
Fourier series approximation,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$



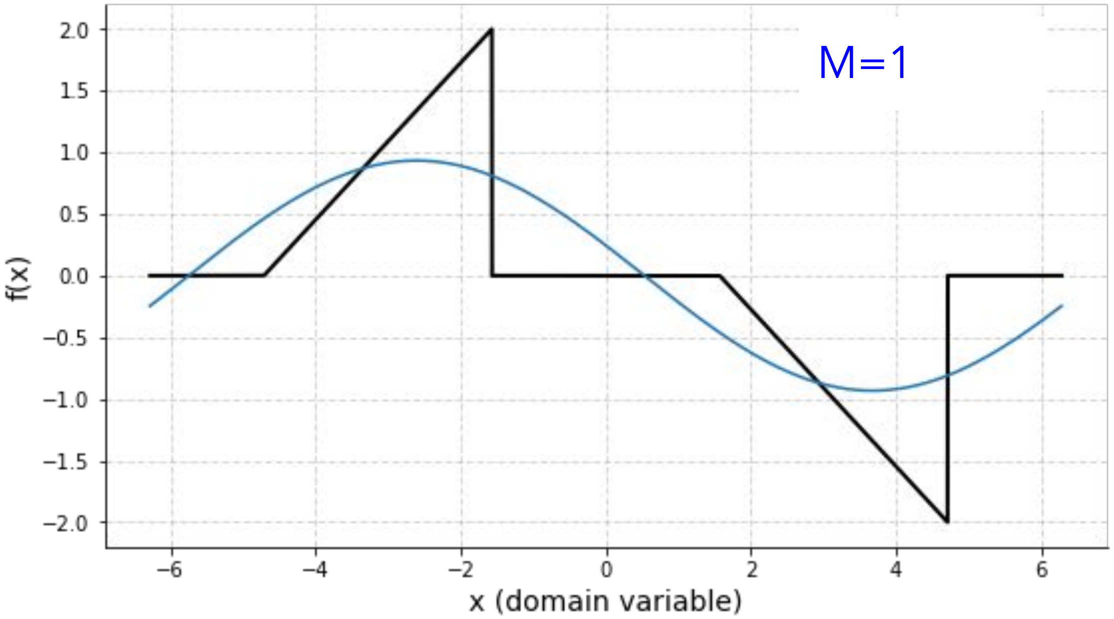


Another example,



Fourier series approximation,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$



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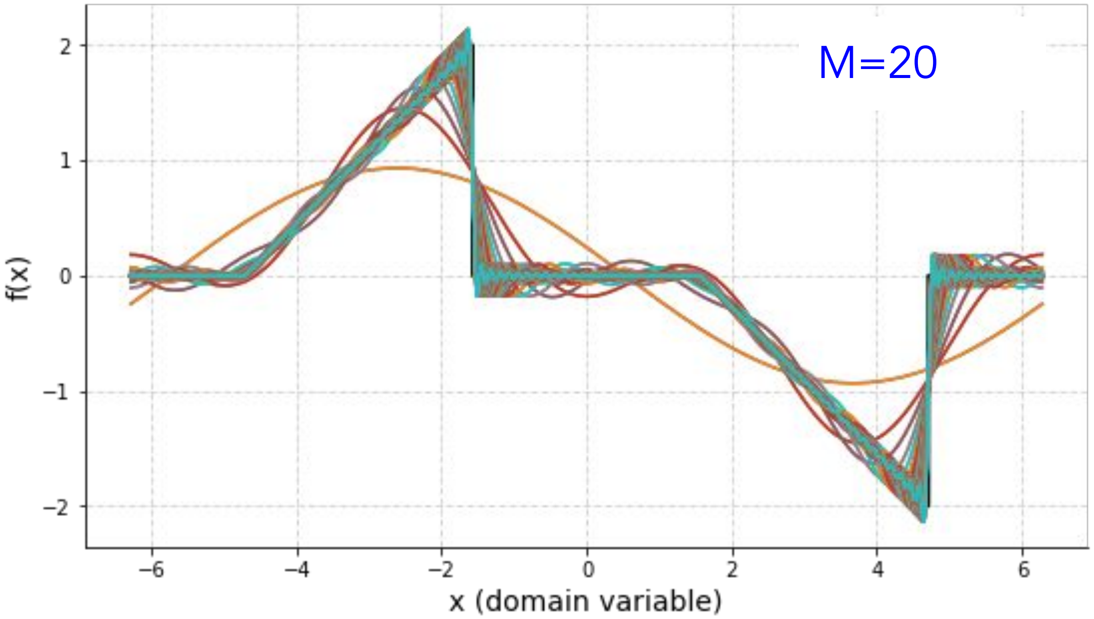
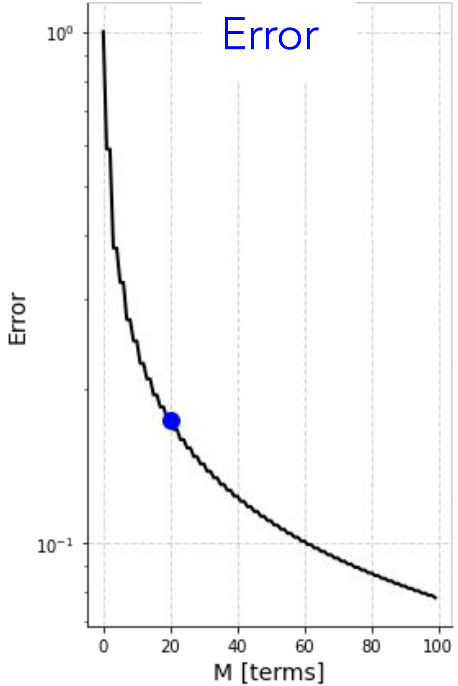


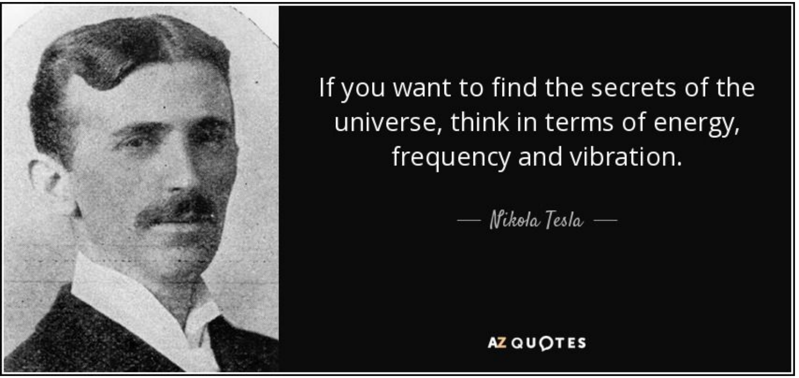


Summary, Fourier series approximation,

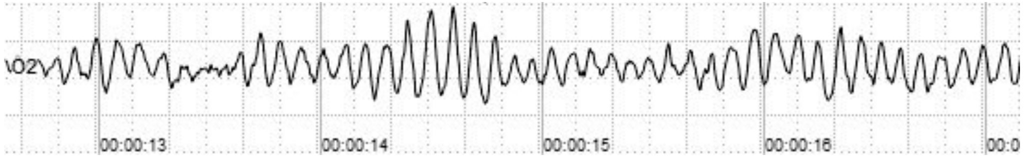
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{M} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

- Can model (or represent) a periodic signal
- Parameters of the model are $\{a_o , a_m , b_m\}$ and M
- Suitable if signal has oscillatory patterns (or fluctuations)

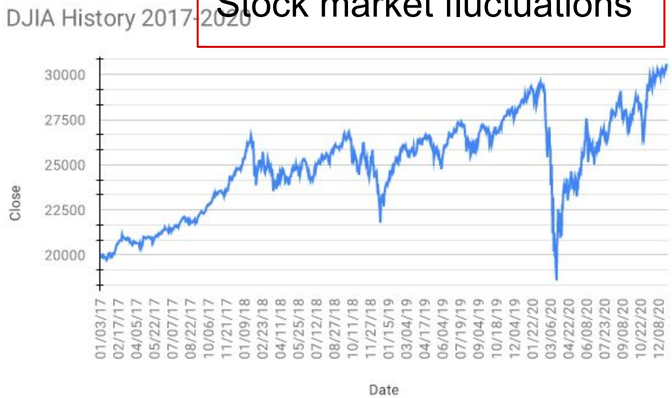
Do periodic signal exist in real-life?



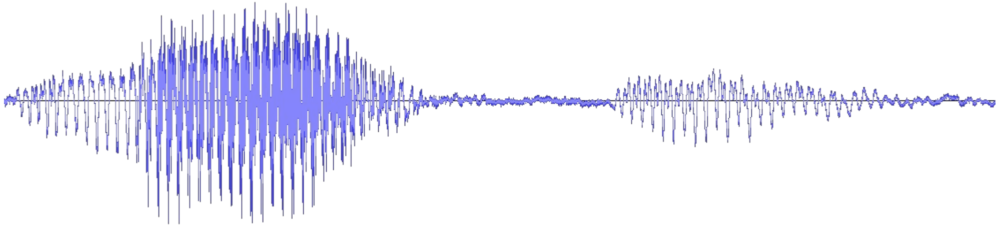
EEG signal (correlate of electrical activity in the brain)



Stock market fluctuations



Air pressure associated with spoken speech utterance



and a lot more! …. We will continue next class.