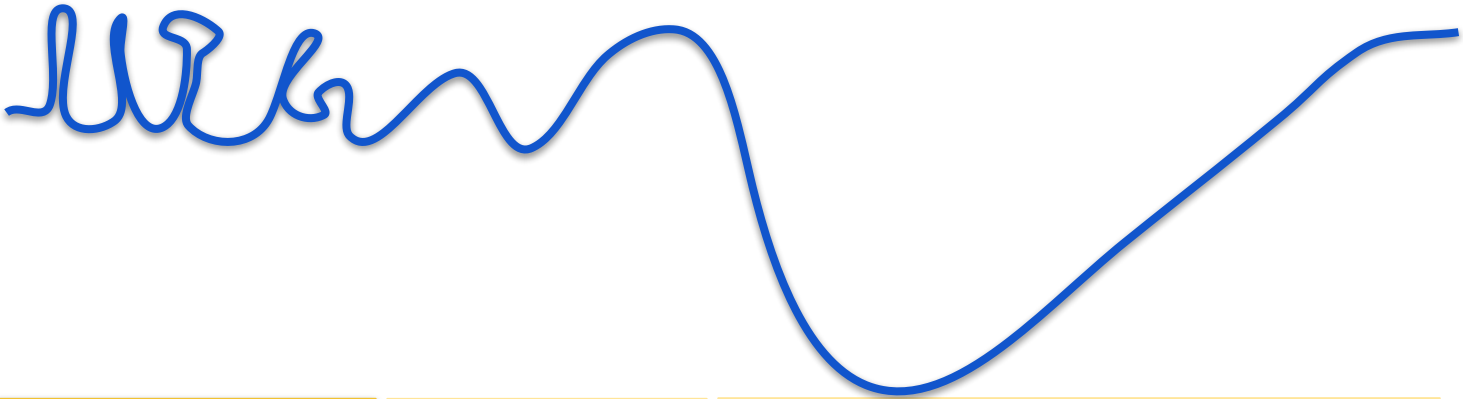
Computing with Signals



DA 623 | Jan - May 2023 | IIT Guwahati

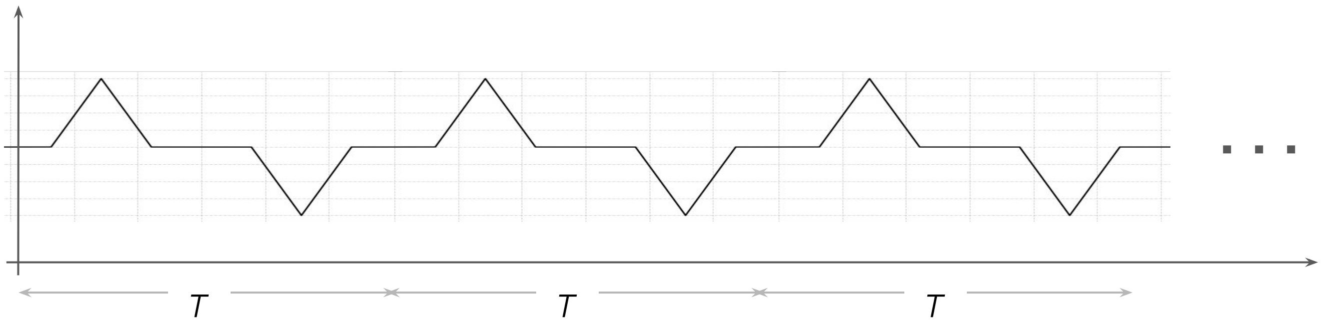
Instructors: Neeraj Sharma

Lecture-08-[23-Jan]

- Polynomial series representation
- Fourier series representation

- Polynomial series representation
- Fourier series representation
- Fourier transform representation

Recap



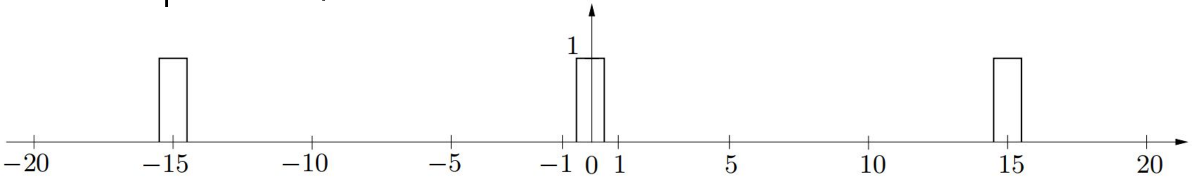
- For a signal with period *T* the Fourier series representation has the following form.

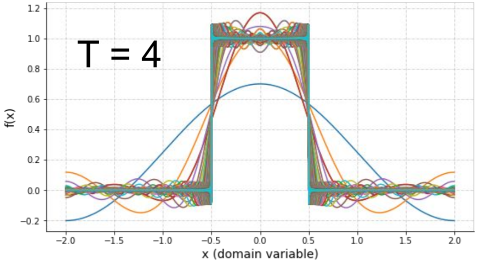
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

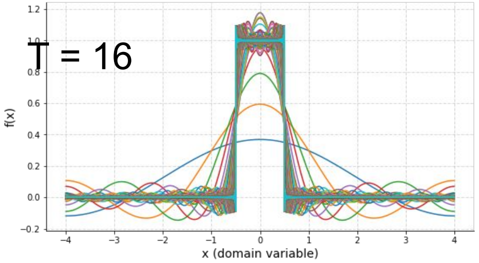
$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt$$

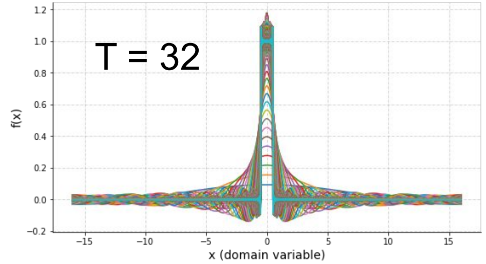
- For a rect(t) signal, periodized with a period T,

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$

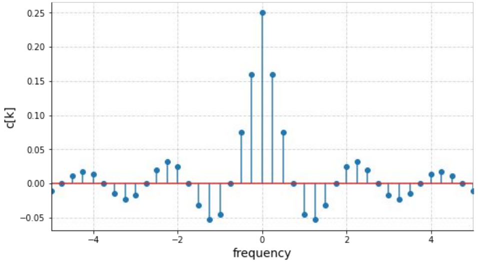


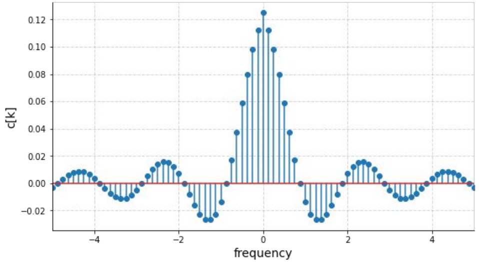


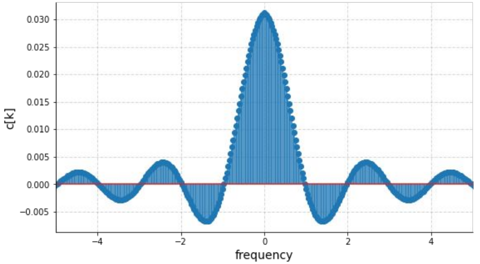




Fourier series coefficients:

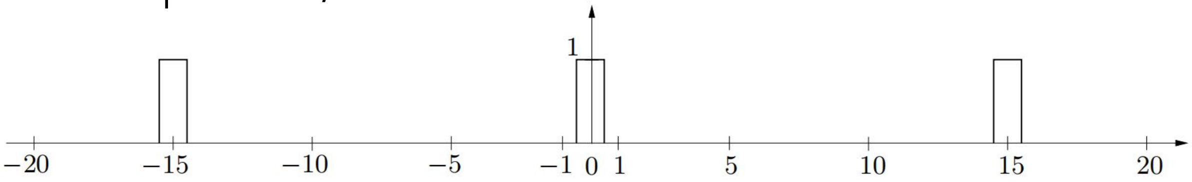


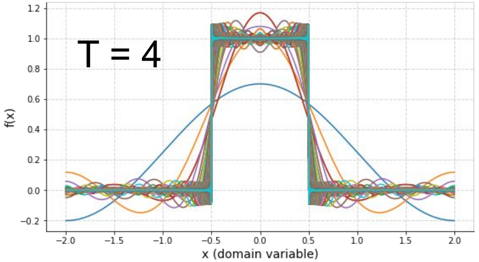


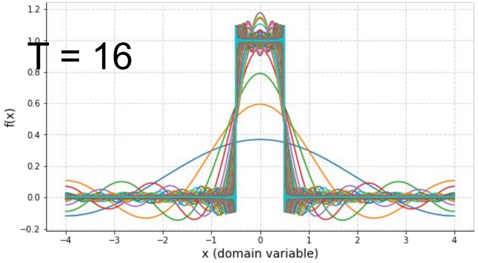


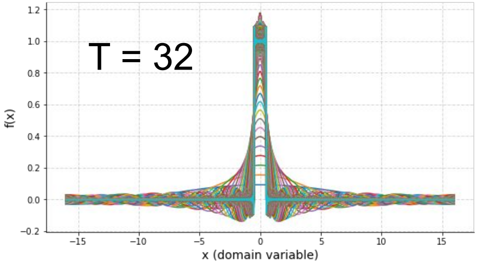
- For a rect(t) signal, periodized with a period T,

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$



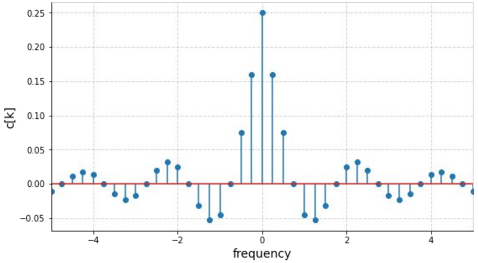


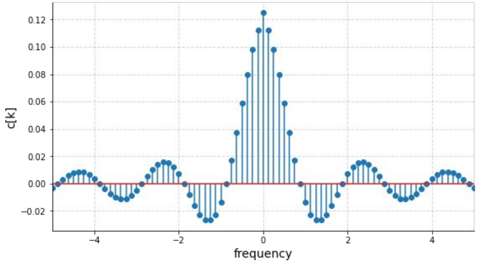


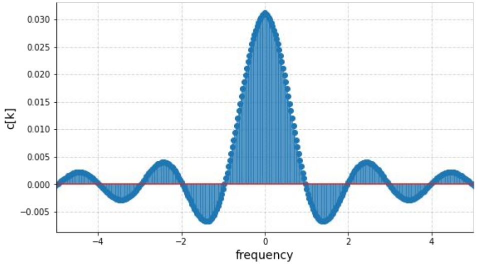


Fourier series coefficients:

Note: The c[k] representation starts getting crowded (or denser) as T increases



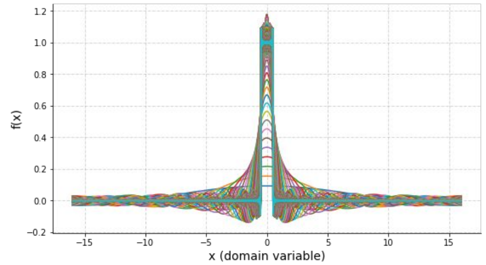




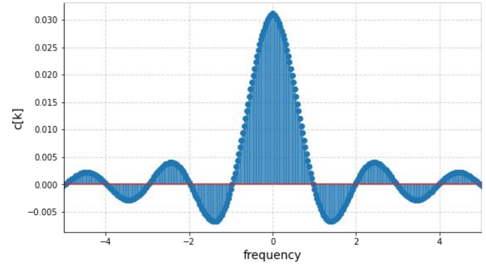
- Computing Fourier series of rect(t) function after periodization with period *T*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T}\Pi(t)\,dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi i n t/T}\cdot 1\,dt$$
$$= \frac{1}{T}\left[\frac{1}{-2\pi i n/T}e^{-2\pi i n t/T}\right]_{t=-1/2}^{t=1/2} = \frac{1}{2\pi i n}\left(e^{\pi i n/T} - e^{-\pi i n/T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$



$$(\text{Transform of periodized }\Pi)\left(\frac{n}{T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$



- Computing Fourier series of rect(t) function after periodization with period *T* , and T going to infinity

$$(\text{Transform of periodized }\Pi)\left(\frac{n}{T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$

$$(\text{Scaled transform of periodized }\Pi)\left(\frac{n}{T}\right) = T\,\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) = \frac{\sin(\pi n/T)}{\pi n/T}$$

$$(\text{Scaled transform of periodized }\Pi)(s) = \frac{\sin \pi s}{\pi s}$$

  - Unlike n/T, as T tends to infinity, *s* represents a continuous variable
  - We now have a representation for an aperiodic signal rect(t), that is, the signal is not periodic as T tends to infinity
  - That’s the Fourier transform

- Ok, we showed what happens when we periodize a rect(t) signal
- Then we made the period go to infinity
- How about any *f(t)* in general

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\, dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi int/T} f(t)\, dt$$

$$\begin{aligned} |c_n| &= \frac{1}{T}\left| \int_{-1/2}^{1/2} e^{-2\pi int/T} f(t)\, dt \right| \\ &\leq \frac{1}{T}\int_{-1/2}^{1/2} |e^{-2\pi int/T}|\, |f(t)|\, dt = \frac{1}{T}\int_{-1/2}^{1/2} |f(t)|\, dt = \frac{A}{T} \end{aligned}$$

$$A = \int_{-1/2}^{1/2} |f(t)|\, dt\,,$$

- Ok, we showed what happens when we periodize a rect(t) function and then make the period infinity.
- How about any *f(t)* in general

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\, dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi int/T} f(t)\, dt$$

$$(\text{Scaled transform of periodized } f)\left(\frac{n}{T}\right) = Tc_n = \int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\, dt$$

In the limit as $T \to \infty$ we replace n/T by s and consider

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\, dt$$

- Fourier Series

The spectrum of a periodic signal is a discrete set of frequencies, possibly an infinite set (when there's a corner) but always a discrete set.

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt$$

- Fourier Transform

By contrast, the Fourier transform of a nonperiodic signal produces a continuous spectrum, or a continuum of frequencies.

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt$$

- Demo
    - Using Audacity (software) and seeing the spectrum of whistling
    - https://www.audacityteam.org/download/
    - Mine spikes around 2100 Hz. Check yours!

Thank you