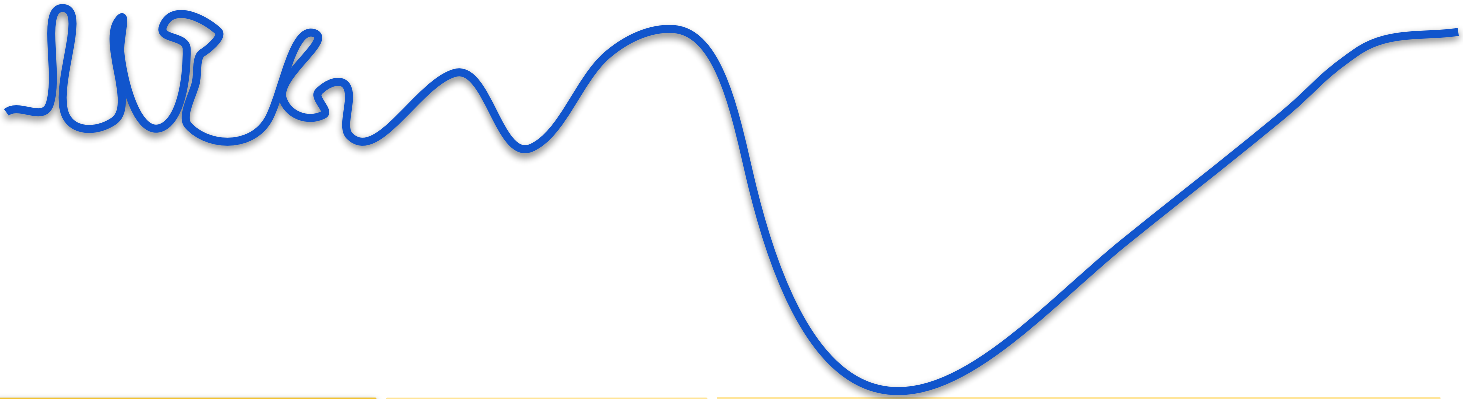
Computing with Signals



DA 623

Jan - May 2023

IIT Guwahati

Instructors: Neeraj Sharma

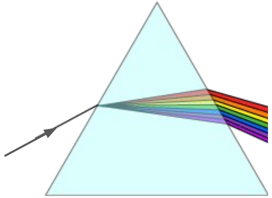
Lecture-09-[27-Jan]

- # Fourier Transform

produces a continuous spectrum, or a continuum of frequencies.

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

Signal $f(t)$ → Fourier Transform → Continuous representation (or spectrum) $\hat{f}(s)$



- Transform signal from t-domain to s-domain
- The two domains can be casually understood as inversely related because we defined s~n/*T* (with *T* tending to infinity)

- Inverse Fourier Transform: Can we get back $\hat{f}(s)$ from $f(t)$?

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt = \frac{1}{T}\int_{-\infty}^{\infty} e^{-2\pi i n t/T} f(t)\, dt$$

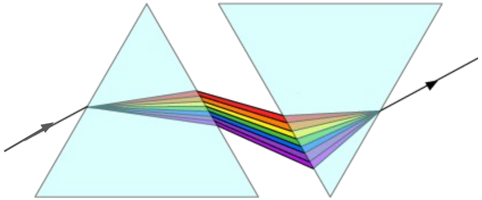
$$= \frac{1}{T}\hat{f}\left(\frac{n}{T}\right) = \frac{1}{T}\hat{f}(s_n)$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\hat{f}(s_n) e^{2\pi i s_n t}$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(s_n) e^{2\pi i s_n t} \Delta s \approx \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i s t}\, ds$$

Yes!

- Fourier Transform and Inverse Transform



$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

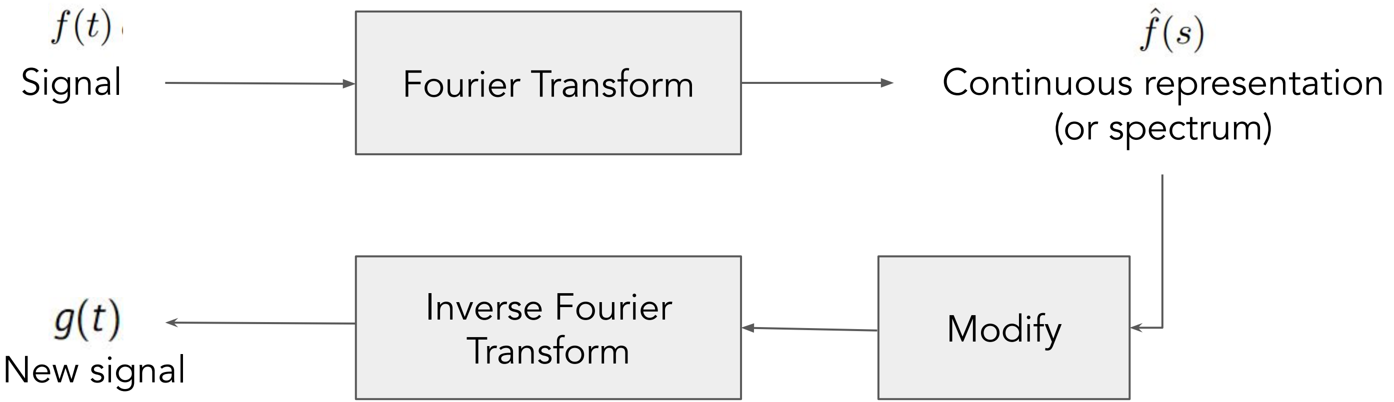
$f(t)$ Signal → Fourier Transform → $\hat{f}(s)$ Continuous representation (or spectrum)

$\hat{f}(s)$ → Inverse Fourier Transform → $f(t)$ Signal

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i s t}\, ds$$

- Modifying the Spectrum

Can we modify the spectrum and get a new (and more useful) signal?



- Modifying the spectrum - But how? Is there a “nice” method?

$$\mathcal{F}[f] := \hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st} dt$$

↑

This will be our new notation for representing Fourier transform

- Few ways to modify a signal

$$m(t) = f(t) + g(t)$$
$$\mathcal{F}[m] = \mathcal{F}[f(t) + g(t)] = \mathcal{F}[f] + \mathcal{F}[g]$$

adding another signal

$$m(t) = \alpha f(t)$$
$$\mathcal{F}[m] = \alpha \mathcal{F}[f]$$

scaling with a constant

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- How about multiplying two spectrums?
  - Use case: spectrum of one signal can be weighted using spectrum of another signal

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi st} g(t)dt \int_{-\infty}^{\infty} e^{-i2\pi sx} f(x)dx$$

  - What is the resulting time domain operation?

- Multiplying two spectrums

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi st} g(t)dt \int_{-\infty}^{\infty} e^{-i2\pi sx} f(x)dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i2\pi s(t+x)} g(t)f(x)dtdx$$

$$(t+x) = u$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i2\pi su} g(u-x)f(x)dudx$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi su} \underbrace{\left(\int_{-\infty}^{\infty} g(u-x)f(x)dx \right)}_{h(u)} du$$

We refer to this fondly by "f is convolved with g"

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi su} h(u)du$$

$$= \mathcal{F}[h]$$

- Multiplying two spectrums

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi su} h(u)du$$

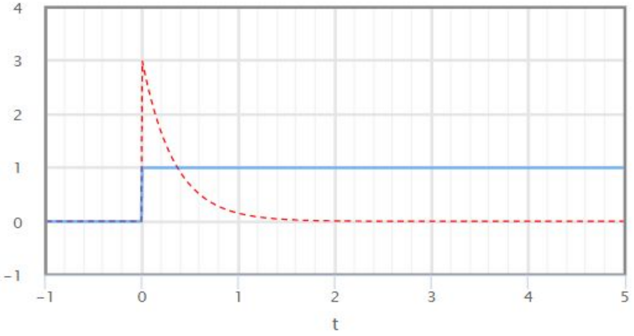
Equivalent operation in time domain

$$\underbrace{\int_{-\infty}^{\infty} g(u-x)f(x)dx}_{h(u)}$$

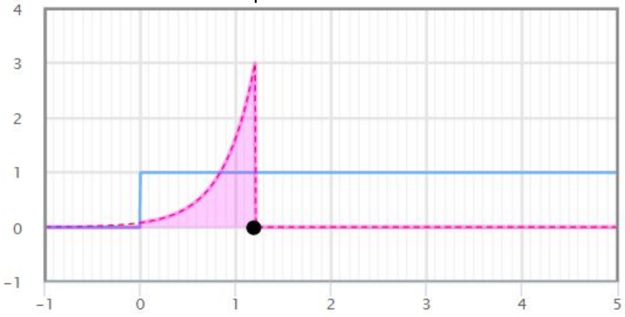
Convolution is born!

Example: https://lpsa.swarthmore.edu/Convolution/CI.html

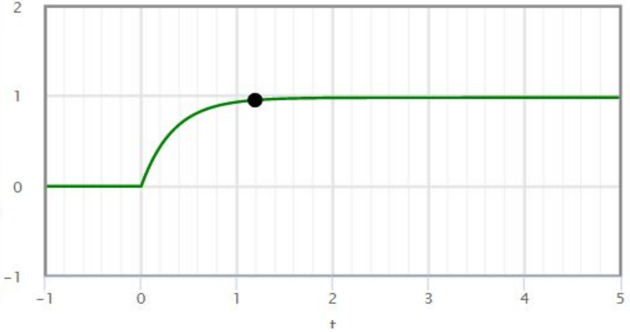
Signals



Step 1: Flip one of them,
Step 2: translate inside



Step 3: Product and Integrate,
Step 4: Go to step 2



- Multiplying two spectrums

$$\mathcal{F}[g]\mathcal{F}[f] = \int_{-\infty}^{\infty} e^{-i2\pi su} h(u)du$$

Equivalent operation in time domain

$$\underbrace{\int_{-\infty}^{\infty} g(u-x)f(x)dx}_{h(u)}$$

- Convolution is a linear operation - nothing fancy but cleaver
- Multiplying two spectrums helps to weight the spectrum of one signal using another
- This weighting operation is useful for:
  - Spectrum enhancement
  - Noise removal
  - Feature extraction

Example: https://dspillustrations.com/pages/posts/misc/convolution-examples-and-the-convolution-integral.html

- Summary

Signal → Model *or* Representation → Processing

- Convolution

- Polynomial series representation
- Fourier series representation
- Fourier transform representation

Thank you