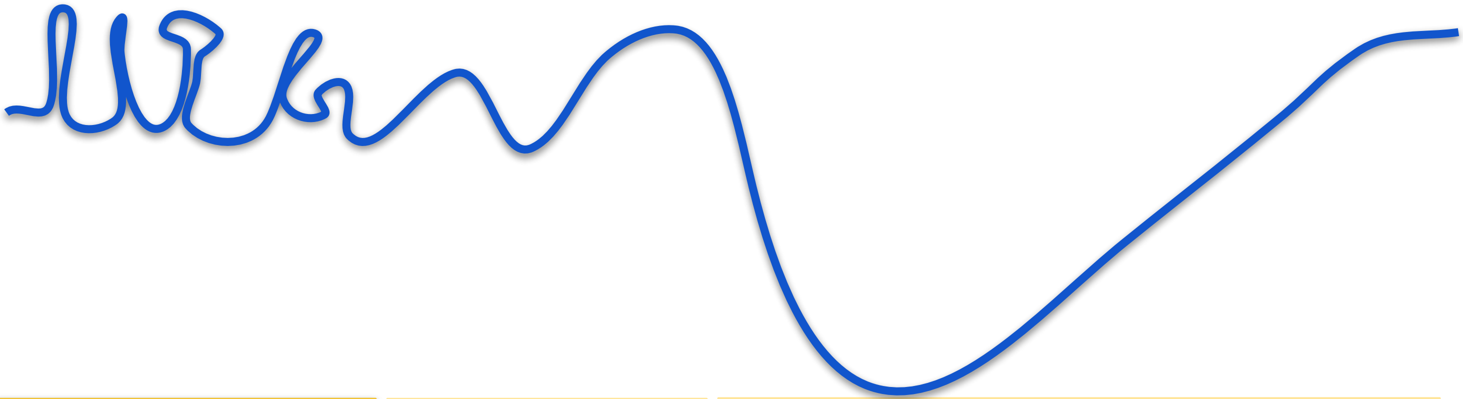
Computing with Signals



DA 623

Jan - May 2023

IIT Guwahati

Instructors: Neeraj Sharma

Lecture-15-[13-Feb]

Sampling and Interpolation (or Reconstruction)

Acknowledgement: Some content is borrowed from the "Lecture Notes for EE 261 - The Fourier Transform and its Applications", by Brad Osgood.

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}, \; t > 0$$

$$\rho_p(x) = \sum_{k=-\infty}^{\infty} \rho(x - kp)$$

$$\rho_p(x) = \sum_{k=-\infty}^{\infty} \rho(x - pk) = \sum_{k=-\infty}^{\infty} \delta(x - kp) * \rho(x)$$

$$= \left(\sum_{k=-\infty}^{\infty} \delta(x - kp) \right) * \rho(x)$$

Periodizing a function $$\rho_p(x) = \sum_{k=-\infty}^{\infty} \rho(x - kp)$$

- Shah function
- Comb function
- Train of Diracs

$$\text{Ш}_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp)$$

$$\rho_p = \text{Ш}_p * \rho \, .$$

$$\text{Ш}_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp) \quad \text{or} \quad \text{Ш}_p = \sum_{k=-\infty}^{\infty} \delta_{kp}$$

$$Ш_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp) \quad \text{or} \quad Ш_p = \sum_{k=-\infty}^{\infty} \delta_{kp}$$

$$\langle Ш_p, \varphi \rangle = \left\langle \sum_{k=-\infty}^{\infty} \delta_{kp}, \varphi \right\rangle = \sum_{k=-\infty}^{\infty} \langle \delta_{kp}, \varphi \rangle = \sum_{k=-\infty}^{\infty} \varphi(kp)$$

The Shah function provides one way to periodizing a function

$$(f * \text{III}_p)(t) = \sum_{k=-\infty}^{\infty} f(t - pk)$$

Of special interest when f is zero for $|t| \geq p/2$ as then,

$$\Pi_p f = f$$

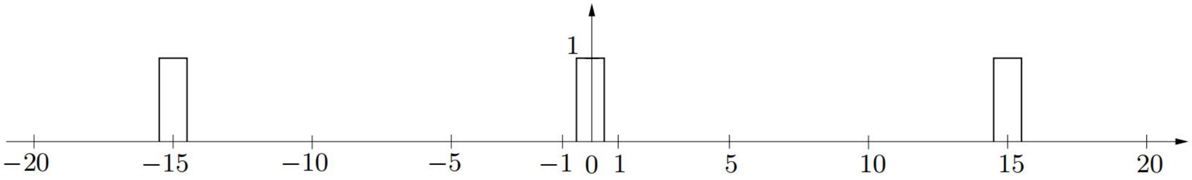
$$f = \Pi_p (f * \text{III}_p)$$

The Shah function provides one way to periodizing a function

$$(f * Ш_p)(t) = \sum_{k=-\infty}^{\infty} f(t - pk)$$

Can you recall we used this approach in earlier classes?

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$



The Shah function provides one way to periodizing a function

$$(f * \text{Ш}_p)(t) = \sum_{k=-\infty}^{\infty} f(t - pk)$$

The Shah function also provides one way sampling a function

$$f(x)\text{Ш}(x) = \sum_{k=-\infty}^{\infty} f(x)\delta(x-k) = \sum_{k=-\infty}^{\infty} f(k)\delta(x-k)$$

“Distributions are what distributions do”, in that fundamentally they are defined by how they act on “genuine” functions, those in S.

The Shah function also provides one way sampling a function

$$f(x)\text{Ш}(x) = \sum_{k=-\infty}^{\infty} f(x)\delta(x-k) = \sum_{k=-\infty}^{\infty} f(k)\delta(x-k)$$

Sampling at arbitrary but regularly spaced points

$$f(x)\text{Ш}_p(x) = \sum_{k=-\infty}^{\infty} f(kp)\delta(x-kp)$$

Scaling the Shah function

$$\text{Ш}_p(x) = \sum_{k=-\infty}^{\infty} \delta(x - kp)$$

$$\text{Ш}(px) = \sum_{k=-\infty}^{\infty} \delta(px - k)$$

Fourier Transform of Shah function

$$\text{III}(x) = \sum_{k=-\infty}^{\infty} \delta(x - k)$$

$$\mathcal{F}\text{III}(s) = \sum_{k=-\infty}^{\infty} e^{-2\pi iks} = \sum_{k=-\infty}^{\infty} e^{2\pi iks} = \text{III}$$

Fourier Transform of Shah function

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$$\boxed{\sum_{n=-N}^{N} e^{2\pi int}}$$

$$\text{III}(x) = \sum_{k=-\infty}^{\infty} \delta(x-k)$$

$$\begin{aligned} \mathcal{F}\text{III}_p(s) &= \frac{1}{p}\mathcal{F}\left(\text{III}\left(\frac{x}{p}\right)\right) \\ &= \frac{1}{p} p\mathcal{F}\text{III}(ps) \quad \text{(stretch theorem)} \\ &= \text{III}(ps) \\ &= \frac{1}{p}\text{III}_{1/p}(s) \end{aligned}$$

$$\mathcal{F}\text{III}(s) = \sum_{k=-\infty}^{\infty} e^{-2\pi iks} = \sum_{k=-\infty}^{\infty} e^{2\pi iks} = \text{III}$$

$$\mathcal{F}f = \Pi_p(\mathcal{F}f * \text{Ш}_p)$$

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$f(t) = \mathcal{F}^{-1}\mathcal{F}f(t)$

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$$
\begin{aligned}
f(t) = \mathcal{F}^{-1}\mathcal{F}f(t) &= \mathcal{F}^{-1}(\Pi_p(\mathcal{F}f * \text{III}_p))(t) \\
&= \mathcal{F}^{-1}\Pi_p(t) * \mathcal{F}^{-1}(\mathcal{F}f * \text{III}_p)(t) \\
&\qquad \text{(taking } \mathcal{F}^{-1} \text{ turns multiplication into convolution)}
\end{aligned}
$$

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&= \mathcal{F}^{-1}\Pi_p(t) * (\mathcal{F}^{-1}\mathcal{F}f(t) \cdot \mathcal{F}^{-1}\text{III}_p(t)) \\
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&\qquad \text{(ditto, except it's convolution turning into multiplication)} \\
&= p \operatorname{sinc} pt * (f(t) \cdot \frac{1}{p}Ш_{1/p}(t))
\end{aligned}
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&\qquad \text{(ditto, except it's convolution turning into multiplication)} \\
&= p \operatorname{sinc} pt * (f(t) \cdot \frac{1}{p}\text{III}_{1/p}(t)) \\
&= \operatorname{sinc} pt * \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right) \delta\left(t - \frac{k}{p}\right) \quad \text{(the sampling property of } \text{III}_p)
\end{aligned}
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&= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right) \operatorname{sinc} pt * \delta\left(t - \frac{k}{p}\right)
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&\qquad \text{(ditto, except it's convolution turning into multiplication)} \\
&= p \operatorname{sinc} pt * \left(f(t) \cdot \frac{1}{p}\text{III}_{1/p}(t)\right) \\
&= \operatorname{sinc} pt * \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right) \delta\left(t - \frac{k}{p}\right) \quad \text{(the sampling property of } \text{III}_p\text{)} \\
&= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right) \operatorname{sinc} pt * \delta\left(t - \frac{k}{p}\right) \\
&= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right) \operatorname{sinc} p\left(t - \frac{k}{p}\right) \quad \text{(the sifting property of } \delta\text{)}
\end{aligned}
$$

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{p}\right) \operatorname{sinc} p \left(t - \frac{k}{p}\right) \quad \text{(the sifting property of } \delta\text{)}$$

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \operatorname{sinc} p(t - t_k) \quad , \quad t_k = \frac{k}{p}$$

- Shannon Sampling Theorem (1949)
- Whittaker Sampling Formula (1915, 1935)

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\, \mathrm{sinc}\, p(t - t_k) \quad , \quad t_k = \frac{k}{p}$$

Discrete samples of f(t)

Interpolating function

Bandwidth of f(t)

Discrete instants at which f(t) was sampled

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\mathrm{sinc}\,p(t - t_k) \quad , \quad t_k = \frac{k}{p}$$

Discrete samples of f(t)

Interpolating function

Bandwidth of f(t)

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Our Discrete world.

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\mathrm{sinc}\,p(t - t_k) \quad , \quad t_k = \frac{k}{p}$$

Discrete samples of f(t)

Interpolating function

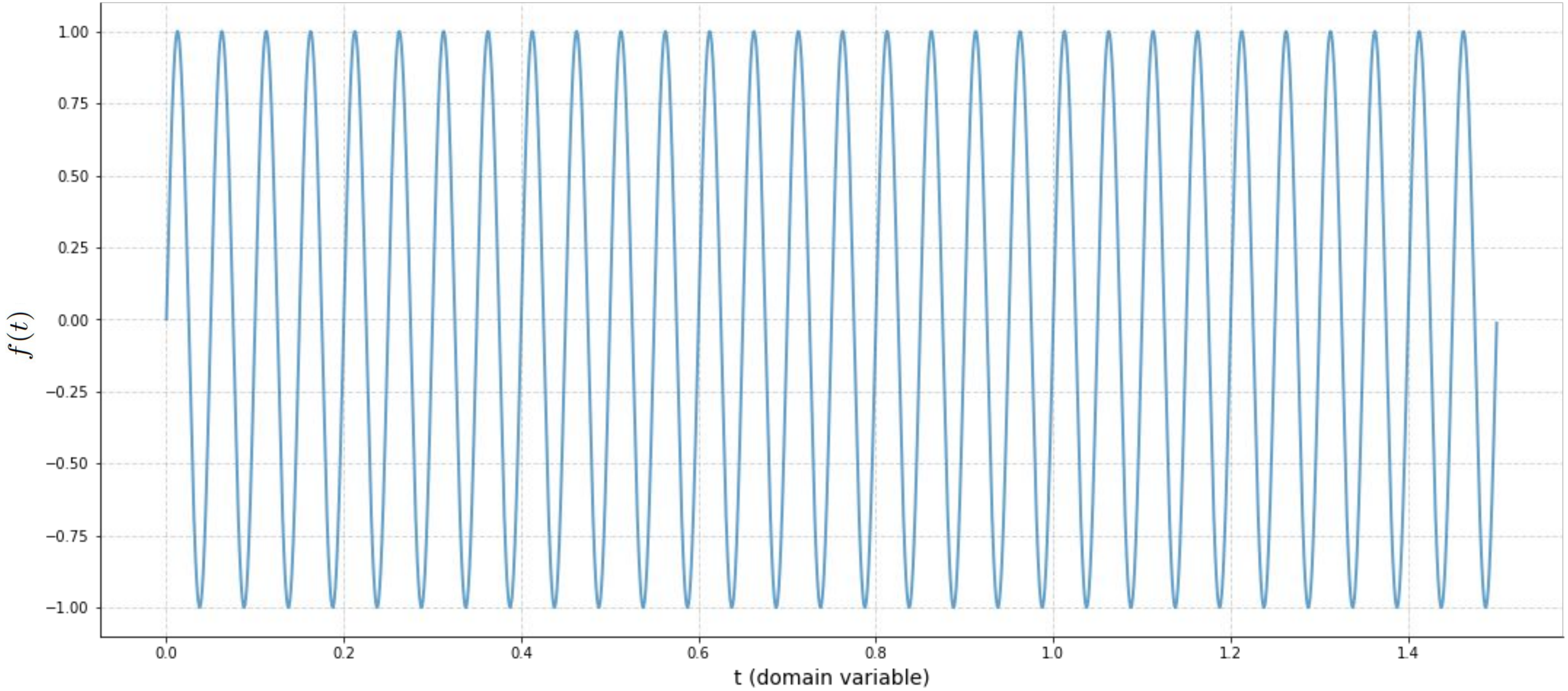
Bandwidth of f(t)

Discrete instants at which f(t) was sampled

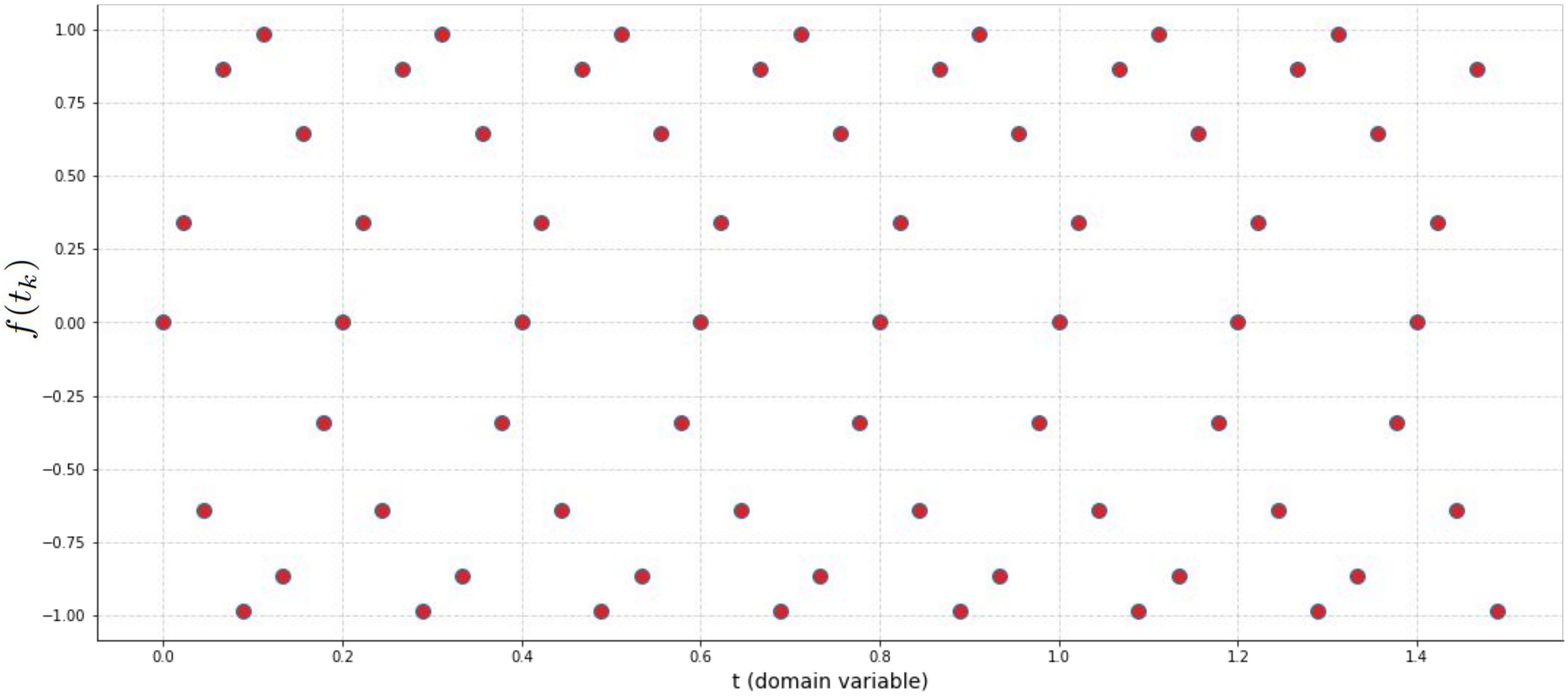
Our Discrete world.

Shannon sampling and reconstruction formula (state above) enables us to switch between the discrete ($f(t_k)$) and continuous world ($f(t)$), without any error!

Example: consider a continuous sine wave signal

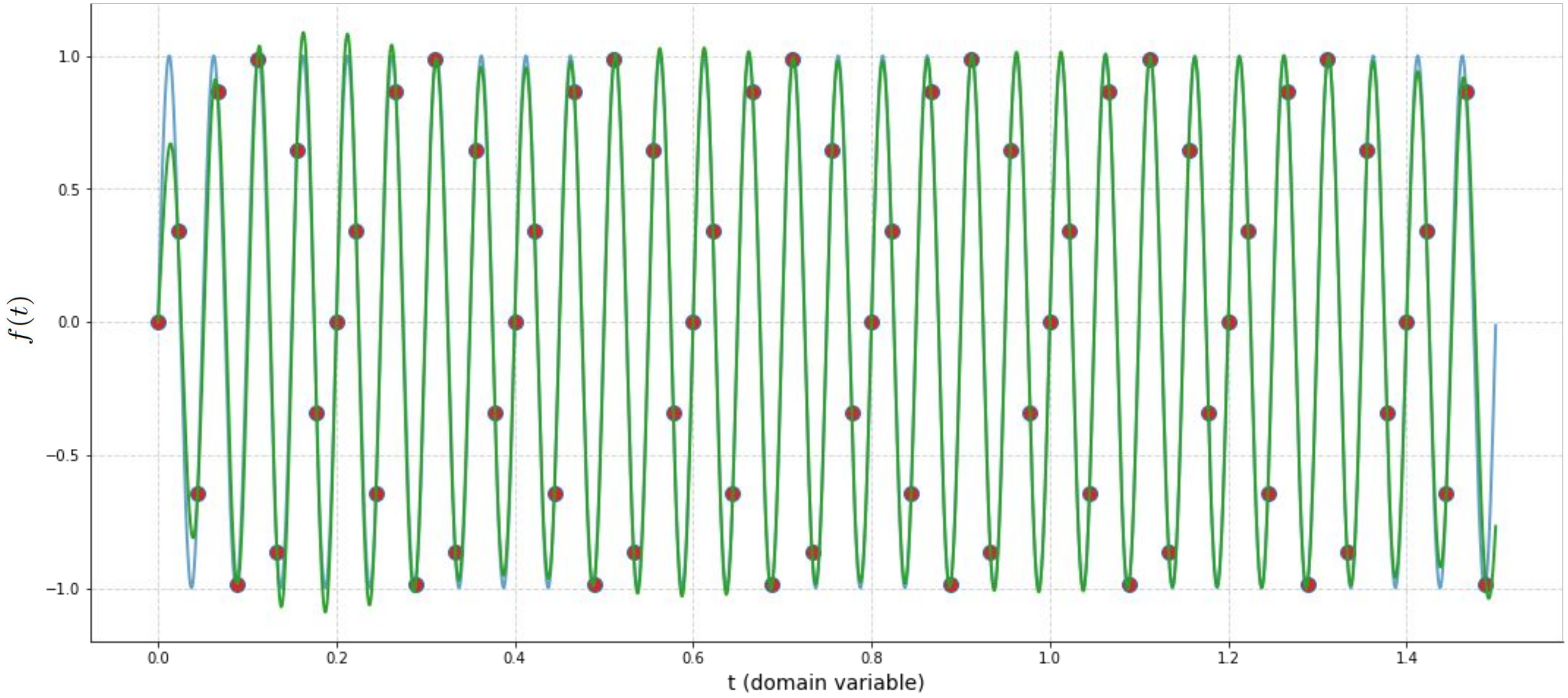


Example: we captured only its samples



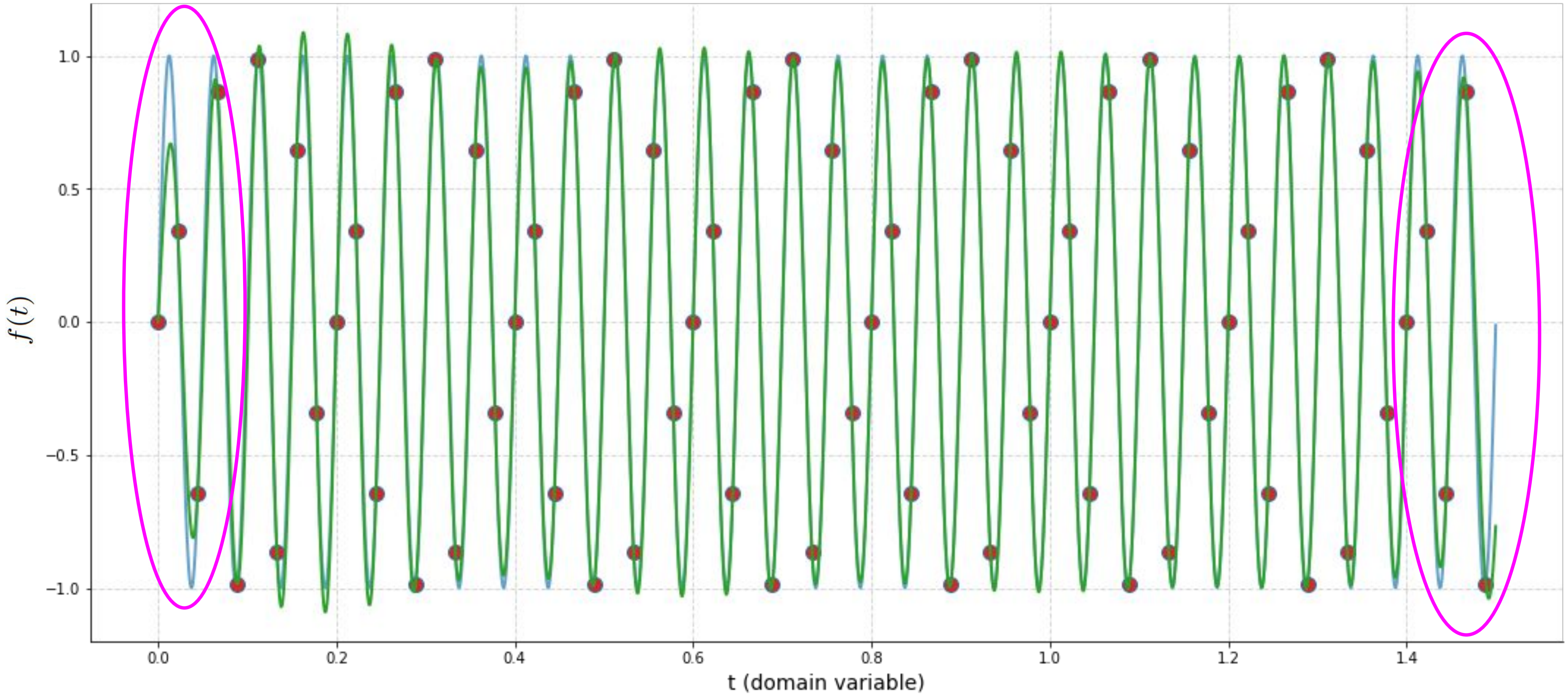
Example: reconstruct using sinc()

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\mathrm{sinc}\,p(t - t_k) \quad , \quad t_k = \frac{k}{p}$$



Example: reconstruct using sinc()

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k)\,\mathrm{sinc}\,p(t - t_k), \qquad t_k = \frac{k}{p}$$



Continued....Thank you!